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EFFECT OF VISCOSITY ON THE CONDITIONS FOR THE FORMATION OF CENTRIFUGAL SOLITONS IN THE TRANSLATIONAL-ROTATIONAL FLOW OF A LIQUID*

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The viscosity of the medium is taken into consideration in deriving an evolution equation describing the propagation of non-linear centrifugal waves along the free surface of a translational-rotational liquid flow. The result is the Burgers-Korteweg-de Vries (BKdV) equation, for which a steady solution is described in the form of a shock wave with soliton oscillators near the front. Estimates are presented for the effect of viscosity on the wave-front structure and the conditions of formation previously predicted by the author /1/ for centrifugal solitons, which play an important role in various atmosphere processes**.

1. Derivation of the evolution equation. The problem of the propagation of centrifugal waves along the free surface of a translational-rotational flow of an ideal incompressible liquid can be reduced /1/ to solving the Laplace equation for the angular component of the vector potential of the velocity field, with non-linear boundary conditions. If viscosity is taken into account, the kinematic boundary condition remains unchanged:

$$v_{r1} = -\left(\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial z} v_{z1}\right) \tag{1.1}$$

but the dynamic condition, obtained by axial projection of the equation of motion for a viscous incompressible liquid, is now

$$\frac{\partial v_{z1}}{\partial t} + v_{z1}\frac{\partial v_{z1}}{\partial z} + \frac{v_\varphi^2}{r_0}\frac{\partial\eta}{\partial z} = \nu\frac{\partial^2 v_{z1}}{\partial z^2} \tag{1.2}$$

Here ν is the kinematic viscosity of the medium; the other notation retains its orginal meanings.

Using the perturbation-theoretical procedure described in /1/, one can derive from (1.1) a BKdV equation for the radial perturbation of the free surface of the flow in variables $\xi = z + c_0 t$, $\tau = -t$:

$$\eta_\tau + {}^3/_2 c_0 h^{-1}\eta\eta_\xi + {}^1/_2 c_0 h^2 \eta_{\xi\xi\xi} + \nu\eta_{\xi\xi} = 0 \tag{1.3}$$

Here we have assumed that the dissipation is weak:

$$\nu/(c_0 l) \ll 1 \quad (c_0 = r_0^{-1} v_\varphi \sqrt{(R^2 - r_0^2)/2} = v_\varphi \sqrt{h/r_0}) \tag{1.4}$$

where c_0 is the propagation velocity of a linear centrifugal wave /2/ along the free surface of a twisted flow of thickness $h = R - r_0 \ll r_0$, and l is the length of the perturbed part of the flow.

When there is no viscosity, (1.3) becomes a Korteweg-de Vries (KdV) equation, which has a one-soliton solution in variables z, t***

$$\eta = \eta_0 \operatorname{sech}^2\frac{z+Vt}{L}, \quad L = 2\sqrt{\frac{h^3}{\eta_0}}, \quad V = c_0\left(1 + \frac{\eta_0}{2h}\right)$$

Dividing Eq.(1.3) by ${}^1/_2 c_0 h^2$, we reduce it to canonical form:

$$\frac{\partial u}{\partial\tau'} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = \nu'\frac{\partial^2 u}{\partial x^2} \tag{1.5}$$

$$(x = -\xi,\ \tau' = -{}^1/_2 c_0 h^2\tau,\ u = 3\eta h^{-3},\ \nu' = 2\nu c_0^{-1} h^{-2})$$

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*Prikl.Matem.Mekhan.,53,6,1035-1037,1989

**BRAZHE R.A., Vortex and soliton phenomena in atmospheric electricity. Ul'yanovsk, 1988. Dep. at VINITI 19.04.1988, 2949-B88.

*** We take this opportunity to rectify an error made in /1/: the coefficient of the non-linear terms in (3.3), (3.4) should be 3/2.

2. ***Steady solution.*** Eq.(1.5) with weak dissipation describes a shock wave with oscillations near the front /3/. To investigate the structure of these oscillations it is proposed /4/ first two write down a periodic solution of the KdV equation, obtained from (1.5) by putting $\nu' = 0$, and then to replace the arbitrary constants appearing in this solution by slowly varying functions of x and τ', for which it is then possible to derive Whitham's averaged equations /5/, generalized to the weakly dissipative case.

We now use the results obtained in /4/. The required solution of the KdV equation is

$$u(x, \tau') = \frac{2a}{s^2} \mathrm{dn}^2 \left[\left(\frac{a}{6s^2} \right)^{1/2} (x - U\tau'), s \right] + U - \frac{2a}{3s^2}(2 - s^2) \tag{2.1}$$

where $\mathrm{dn}(y, s)$ is the Jacobi function of modulus s, a is the amplitude of the oscillations, and U defines the phase velocity of the wave in a reference system associated with the variables x, τ'.

In the reference system $X = x - U\tau'$, in which the wave has a steady profile, the mean value of the wave function $\langle u \rangle$, the amplitude a and wavelength λ may be written as follows /4/:

$$\langle u \rangle = {}^1/_2 [1 - (2 - s^2 - 3\mathrm{E}/\mathrm{K}) f^{-1/2}] u^-, \quad a = {}^3/_4 s^2 f^{-1/2} u^-$$
$$\lambda = 4\sqrt{2} f^{1/4} \mathrm{K} (u^-)^{-1/2}, \quad f(s) = (1 - s^2 + s^4) \tag{2.2}$$

where K and E are complete elliptic integrals of the first and second kinds of modulus s, u^- the size of the jump at the shock-wave front, $U = {}^1/_2 u^-$.

The value of s is determined by numerical solution of the equation

$$\nu' (X - X_0) = F(s), \quad F(s) = \ln [f(s)\mathrm{E} - (1 - s^2)(1 - s^2/2)\mathrm{K}] - {}^5/_4 \ln f(s) \tag{2.3}$$

where X_0 is the coordinate of the onset of the wave, chosen arbitrarily.

Substituting (2.2) and (2.3) into (2.1), one can uniquely determine the oscillatory structure of the wave in a weakly dissipative medium. As the dissipation increases, the profile of the shock wave front becomes monotonic. The solution of the BKdV equation for this case was studied in /6/, using Bäcklund transformations.

It follows from (2.3) that far from the leading front of the centrifugal wave $(X \ll X_0)$ $s \ll 1$, and the function $\mathrm{dn}(y, s)$ is approximately a superposition of harmonic functions /7/. Near the front $(X \approx X_0)$; however $s \to 1$ and $\mathrm{dn}(y, s) \to \mathrm{sech}\, y$, i.e., the oscillations take the form of a sequence of solitons at distance $\lambda_s = \lambda$ from one another, where λ has the value indicated in (2.2).

Putting $X_0 - X = \lambda_s$ in (2.3) and going back to the original notation, one can find the minimum rotational velocity of the flow for which soliton-like oscillations can form at the leading edge of the shock wave:

$$v_{\varphi\,\min} = \frac{8\nu}{h} \sqrt{\frac{2r_0}{3\eta^-}} \frac{f(s)^{1/4}\mathrm{K}}{F(s)}, \quad \eta^- = \frac{h^3 u^-}{3}$$

where η^- is the size of the jump in the radial shift of the free surface at the shock wave front.

We end with a few estimates, putting $r_0 = 10^{-1}$ m, $h = 10^{-2}$ m, and $\eta^- = 10^{-3}$ m. For water at room temperature $\nu = 1.05 \times 10^{-6}$ m^2/sec, which gives $v_{\varphi\min} \approx 10.5$ m/sec. For a less viscous liquid, centrifugal solitons form more easily. In the case of acetone, for example ($\nu = 4.26 \times 10^{-7}$ m^2/sec), $v_{\varphi\min} \approx 4.2$ m/sec.

These results show that the formation of centrifugal solitons in translational-rotational flow in real liquids in quite possible.

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